New Versions of Some Classical Stochastic Inequalities

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We start with a quotation from the preface of V. V. Petrov's book titled *Limit Theorems of Probability* (Oxford, 1995): "Limit theorems of probability theory form an evergreen field of probability theory. Its methods and results continue to have great influence on other fields of probability theory, mathematical statistics, and their applications."

Stochastic inequalities are a crucial tool for establishing probability limit theorems and for advancing statistical theory, and they also have an intrinsic interest as well.

Some Classical Stochastic Inequalities

In this talk new versions of the classical Lévy, Ottaviani, and Hoffmann-Jørgensen inequalities are presented for a sequence of real-valued/B-valued random variables.

Let $X_1, ..., X_n$ be independent real-valued random variables with $S_0 \equiv 0$ and $S_k = X_1 + \cdots + X_k$, $1 \leq k \leq n$. Let $\mathrm{m}(X)$ be the median of the random variable X.

Some Classical Stochastic Inequalities

Theorem A (*The classical Lévy inequalities*) For every real x,

$$\mathbb{P}\left(\max_{1\leq k\leq n}\left(S_k - \mathsf{m}(S_k - S_n)\right) \geq x\right) \leq 2\mathbb{P}\left(S_n \geq x\right).$$

For every $x \geq 0$ we have

$$\mathbb{P}\left(\max_{1\leq k\leq n}\left|S_k - \mathsf{m}(S_k - S_n)\right| \geq x\right) \leq 2\mathbb{P}\left(|S_n| \geq x\right).$$

Some Classical Stochastic Inequalities

If $X_1, ..., X_n$ are independent symmetric real-valued random variables, then, for every real x,

$$\mathbb{P}\left(\max_{1\le k\le n} S_k \ge x\right) \le 2\mathbb{P}\left(S_n \ge x\right)$$

and, for every $x \ge 0$,

$$\mathbb{P}\left(\max_{1\le k\le n} |S_k| \ge x\right) \le 2\mathbb{P}\left(|S_n| \ge x\right)$$

and

$$\mathbb{P}\left(\max_{1\le k\le n} |X_k| \ge x\right) \le 2\mathbb{P}\left(|S_n| \ge x\right).$$

The classical Lévy inequalities were obtained by P. Lévy (1937). There is also a generalization of them to martingales (see Loève, 1963).

Some Classical Stochastic Inequalities

Theorem B (*The classical Ottaviani inequality* (see Chow and Teicher, 1988, p. 74)) For every $x \geq 0$, we have

$$\mathbb{P}\left(\max_{1\leq k\leq n} |S_k| \geq 2x\right) \leq \frac{\mathbb{P}\left(|S_n| \geq x\right)}{\min_{0\leq j\leq n} \mathbb{P}\left(|S_n - S_j| \leq x\right)}.$$

In particular, if for some $\delta \geq 0$,

$$\max_{0\leq j\leq n} \mathbb{P}\left(|S_n - S_j| \geq \delta\right) \leq \frac{1}{2},$$

then for every $x \geq \delta$, we have

$$\mathbb{P}\left(\max_{1\leq k\leq n} |S_k| \geq 2x\right) \leq 2\mathbb{P}\left(|S_n| \geq x\right).$$

Some Classical Stochastic Inequalities

Let $(\mathbf{B}, \|\cdot\|)$ be a real separable Banach space equipped with its Borel σ-algebra $\mathcal{B}$ (= the σ-algebra generated by the class of open subsets of $\mathbf{B}$ determined by $\|\cdot\|$).

Let $\{X_n;\ n \geq 1\}$ be a sequence of independent **B**-valued random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Let $\mathbf{B}^\infty = \mathbf{B} \times \mathbf{B} \times \mathbf{B} \times \cdots$.

Let $q: \ \mathbf{B}^\infty \to \overline{\mathbb{R}}_+ = [0, \infty]$ be a measurable function. Write

$$S_n = q\left(X_1, \cdots, X_n, 0, \cdots\right), \quad Y_n = q\left(0, \cdots, 0, X_n, 0, \cdots\right),$$

$$M_n = \max_{1 \leq j \leq n} S_j, \quad N_n = \max_{1 \leq j \leq n} Y_j, \quad n \geq 1$$

and

$$M = \sup_{n \geq 1} S_n, \quad N = \sup_{n \geq 1} Y_n.$$

Some Classical Stochastic Inequalities

We say that q is a quasiconvex function if, for all $\mathbf{x}, \mathbf{y} \in \mathbf{B}^{\infty}$,

$$q\left(t\mathbf{x} + (1-t)\mathbf{y}\right) \leq \max\left(q(\mathbf{x}),\ q(\mathbf{y})\right) \quad \text{whenever} \quad 0 \leq t \leq 1. \tag{1}$$

We say that q is a subadditive function if, for all $\mathbf{x}, \mathbf{y} \in \mathbf{B}^{\infty}$,

$$q\left(\mathbf{x} + \mathbf{y}\right) \leq q(\mathbf{x}) + q(\mathbf{y}). \tag{2}$$

The following Theorems C and D were obtained by Jorgen Hoffmann-Jørgensen (1974) where Theorem C is version of the classical Lévy inequality in a Banach space setting and the results presented in Theorem D are what we call the classical Hoffmann-Jørgensen inequalities.

Theorem C Let $\{X_n;\ n \geq 1\}$ be a sequence of independent symmetric **B**-valued random variables. Let $q:\ \mathbf{B}^\infty \to \overline{\mathbb{R}}_+ = [0, \infty]$ be a measurable quasiconvex function (i.e., (1) holds). Then for every $n \geq 1$ and every $t \geq 0$, we have

$$\mathbb{P}\left(M_n > t\right) \leq 2\mathbb{P}\left(S_n > t\right)$$

and

$$\mathbb{P}\left(N_n > t\right) \leq 2\mathbb{P}\left(S_n > t\right).$$

Moreover if $S_n \to S$ in law, then for every $t \geq 0$, we have

$$\mathbb{P}(M > t) \leq 2\mathbb{P}\left(S > t\right)$$

and

$$\mathbb{P}(N > t) \leq 2\mathbb{P}\left(S > t\right).$$

Theorem D Let $\{X_n;\ n \geq 1\}$ be a sequence of independent symmetric **B**-valued random variables. Let $q:\ \mathbf{B}^{\infty} \to \overline{\mathbb{R}}_+ = [0, \infty]$ be a measurable quasiconvex and subadditive function (i.e., the conditions (1) and (2) hold). Then for all $n \geq 1$ and all nonnegative real numbers s, t, and u, we have

$$\begin{aligned}\mathbb{P}\left(S_n > s+t+u\right) &\leq \mathbb{P}\left(N_n > s\right) + 2\mathbb{P}\left(S_n > u\right)\mathbb{P}\left(M_n > t\right) \\ &\leq \mathbb{P}\left(N_n > s\right) + 4\mathbb{P}\left(S_n > u\right)\mathbb{P}\left(S_n > t\right),\end{aligned}$$

$$\mathbb{P}\left(M_n > s+t+u\right) \leq 2\mathbb{P}\left(N_n > s\right) + 8\mathbb{P}\left(S_n > u\right)\mathbb{P}\left(S_n > t\right),$$

and

$$\mathbb{P}\left(M > s+t+u\right) \leq 2\mathbb{P}\left(N > s\right) + 4\mathbb{P}\left(M > u\right)\mathbb{P}\left(M > t\right).$$

Some Classical Stochastic Inequalities

The classical Hoffmann-Jørgensen inequalities are very powerful in the investigation of general problems on the limit theorems of sums of independent real-valued/B-valued random variables.

During the last thirty-five years, the classical Hoffmann-Jørgensen inequalities have been improved and generalized extensively by some authors; see, for example,

S. Ghosal and T. Chandra (J. Theor. Probab. 11, (1998), 621-631)

M. Klass and K. Nowicki (Ann. Probab. 28, (2000), 851-862)

P. Hitczenko and S. Montgomery-Smith (Ann. Probab. 29, (2001), 447-466)

We now start to present new versions of the classical Lévy, Ottaviani, and Hoffmann-Jørgensen inequalities.

The first theorem is a new version of Lévy's inequality.

Theorem 1 Let $\{X_n;\ n \geq 1\}$ be a sequence of independent symmetric **B**-valued random variables. Let $q:\ \mathbf{B}^{\infty} \to \overline{\mathbb{R}}_{+} = [0, \infty]$ be a measurable function such that for all $\mathbf{x}, \mathbf{y} \in \mathbf{B}^{\infty}$,

$$q\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right) \leq \alpha \max\left(q(\mathbf{x}),\ q(\mathbf{y})\right), \tag{3}$$

where $1 \leq \alpha < \infty$ is a constant, depending only on the function q.

New Versions

Then for every $t \geq 0$, we have

$$\mathbb{P}\left(M_n > t\right) \leq 2\mathbb{P}\left(S_n > \frac{t}{\alpha}\right) \tag{4}$$

and

$$\mathbb{P}\left(N_n > t\right) \leq 2\mathbb{P}\left(S_n > \frac{t}{\alpha}\right). \tag{5}$$

Moreover if $S_n \to S$ in law, then for every $t \geq 0$, we have

$$\mathbb{P}(M > t) \leq 2\mathbb{P}\left(S > \frac{t}{\alpha}\right) \tag{6}$$

and

$$\mathbb{P}(N > t) \leq 2\mathbb{P}\left(S > \frac{t}{\alpha}\right). \tag{7}$$

Remark 1 Clearly Theorem C follows from Theorem 1 if $\alpha = 1$.

New Versions

The second theorem is a new version of Ottaviani's inequality.

Theorem 2 Let $\{X_n;\ n \geq 1\}$ be a sequence of independent **B**-valued random variables. Let $q:\ \mathbf{B}^{\infty} \to \overline{\mathbb{R}}_+ = [0, \infty]$ be a measurable function such that for all $\mathbf{x}, \mathbf{y} \in \mathbf{B}^{\infty}$,

$$q(\mathbf{x} + \mathbf{y}) \leq \beta \left(q(\mathbf{x}) + q(\mathbf{y}) \right), \tag{8}$$

where $1 \leq \beta < \infty$ is a constant, depending only on the function q. Then for every $n \geq 1$ and all $t \geq 0, u \geq 0$, we have

$$\mathbb{P}\left(M_n > t + u\right) \leq \frac{\mathbb{P}\left(S_n > \frac{t}{\beta}\right)}{\min_{1 \leq k \leq n-1} \mathbb{P}\left(D_{n,k+1} \leq \frac{u}{\beta}\right)},$$

where

$$D_{n,j} = q\left(0, \cdots, 0, -X_{j+1}, \cdots, -X_n, 0, \cdots\right),\ j = 1, 2, ..., n-1.$$

In particular, if for some $\delta \geq 0$,

$$\max_{1 \leq k \leq n-1} \mathbb{P}\left(D_{n,k+1} > \frac{\delta}{\beta}\right) \leq \frac{1}{2},$$

then for every $t \geq \delta$, we have

$$\mathbb{P}\left(M_n > 2t\right) \leq 2\mathbb{P}\left(S_n > \frac{t}{\beta}\right).$$

Remark 2 Clearly Theorem B follows from Theorem 2 if

$$q\left(x_1, x_2, \cdots, x_n, \cdots\right) = \left|\sum_{k=1}^{n} x_i\right|, \quad x_i \in (-\infty, \infty), \ i = 1, 2, ..., n.$$

New Versions

The third theorem is a set of new versions of the classical Hoffmann-Jøgensen inequalities (i.e., Theorem D above).

Theorem 3 Let $\{X_n;\ n \geq 1\}$ be a sequence of independent symmetric **B**-valued random variables. Let $q:\ \mathbf{B}^{\infty} \to \overline{\mathbb{R}}_{+} = [0, \infty]$ be a measurable function satisfying conditions (3) and (8). Then for all nonnegative real numbers s, t, and u, we have

$$\begin{aligned} &\mathbb{P}\left(S_n > s + t + u\right) \\ &\leq \mathbb{P}\left(N_n > \frac{s}{\beta^2}\right) + 2\mathbb{P}\left(S_n > \frac{u}{\alpha\beta}\right)\mathbb{P}\left(M_n > \frac{t}{\beta^2}\right) \\ &\leq \mathbb{P}\left(N_n > \frac{s}{\beta^2}\right) + 4\mathbb{P}\left(S_n > \frac{u}{\alpha\beta}\right)\mathbb{P}\left(S_n > \frac{t}{\alpha\beta^2}\right), \end{aligned}$$

$$\mathbb{P}\left(M_n > s + t + u\right)$$

$$\leq 2\mathbb{P}\left(N_n > \frac{s}{\alpha\beta^2}\right) + 8\mathbb{P}\left(S_n > \frac{u}{\alpha^2\beta}\right)\mathbb{P}\left(S_n > \frac{t}{\alpha^2\beta^2}\right),$$

and

$$\mathbb{P}\left(M > s + t + u\right)$$

$$\leq 2\mathbb{P}\left(N > \frac{s}{\alpha\beta^2}\right) + 4\mathbb{P}\left(M > \frac{u}{\alpha^2\beta}\right)\mathbb{P}\left(M > \frac{t}{\alpha\beta^2}\right).$$

Remark 3 Clearly Theorem D follows from Theorem 3 if $\alpha = 1$ and $\beta = 1$.

Proof of Theorme 1 We first prove (4). We consider the stopping time

$$T_1 = \inf \{j \geq 1 : \; S_j > t\} \quad (\inf \emptyset = \infty).$$

For $1 \leq j \leq n$, (3) implies that

$$S_j \leq \alpha \max (S_n, \; S_{n,j})$$

where

$$S_{n,j} = q\left(X_1, \cdots, X_j, -X_{j+1}, \cdots, -X_n, 0, \cdots\right), \; j = 1, 2, ..., n.$$

We thus have that, for $1 \leq j \leq n$

$$\{T_1 = j\} \subseteq \left\{S_n > \frac{t}{\alpha}\right\} \bigcup \left\{S_{n,j} > \frac{t}{\alpha}\right\}$$

and hence that

$$\mathbb{P}(T_1 = j) \leq \mathbb{P}\left(T_1 = j, S_n > \frac{t}{\alpha}\right) + \mathbb{P}\left(T_1 = j, S_{n,j} > \frac{t}{\alpha}\right).$$

Since $\{X_n;\ n \geq 1\}$ is a sequence of independent symmetric **B**-valued random variables, it is easy to see that

$$\mathbb{P}\left(T_1 = j, S_{n,j} > \frac{t}{\alpha}\right) = \mathbb{P}\left(T_1 = j, S_n > \frac{t}{\alpha}\right), \quad j = 1, ..., n.$$

We now have that

$$\begin{aligned}
\mathbb{P}\left(T_1 \leq n\right) &= \sum_{j=1}^{n} \mathbb{P}(T_1 = j) \\
&\leq 2\sum_{j=1}^{n} \mathbb{P}\left(T_1 = j, S_n > \frac{t}{\alpha}\right) \\
&\leq 2\mathbb{P}\left(S_n > \frac{t}{\alpha}\right).
\end{aligned}$$

Note that for all $n \geq 1$ and all $t \geq 0$,

$$\{M_n > t\} = \{T_1 \leq n\}.$$

Thus the conclusion (4) follows.
For proving (5), we consider the stopping time

$$T_2 = \inf\{j \geq 1: \; Y_j > t\} \quad (\inf \emptyset = \infty).$$

For $1 \leq j \leq n$, (3) implies that

$$Y_j \leq \alpha \max\left(S_n, \; S_n^{(j)}\right),$$

where

$$S_n^{(j)} = q\left(-X_1, \cdots, -X_{j-1}, X_j, -X_{j+1}, \cdots, -X_n, 0, \cdots\right), \; j = 1, 2, ..., n$$

We thus have that, for $1 \leq j \leq n$

$$\{T_2 = j\} \subseteq \left\{S_n > \frac{t}{\alpha}\right\} \bigcup \left\{S_n^{(j)} > \frac{t}{\alpha}\right\}$$

and hence that

$$\mathbb{P}(T_2 = j) \leq \mathbb{P}\left(T_2 = j, S_n > \frac{t}{\alpha}\right) + \mathbb{P}\left(T_2 = j, S_n^{(j)} > \frac{t}{\alpha}\right).$$

Since $\{X_n;\ n \geq 1\}$ is a sequence of independent symmetric **B**-valued random variables, it is easy to see that

$$\mathbb{P}\left(T_2 = j, S_n^{(j)} > \frac{t}{\alpha}\right) = \mathbb{P}\left(T_2 = j, S_n > \frac{t}{\alpha}\right), \quad j = 1, ..., n.$$

We now have that

$$\begin{aligned} \mathbb{P}\left(T_2 \leq n\right) &= \sum_{j=1}^{n} \mathbb{P}(T_2 = j) \\ &\leq 2\sum_{j=1}^{n} \mathbb{P}\left(T_2 = j, S_n > \frac{t}{\alpha}\right) \\ &\leq 2\mathbb{P}\left(S_n > \frac{t}{\alpha}\right). \end{aligned}$$

Note that for all $n \geq 1$ and all $t \geq 0$,

$$\{N_n > t\} = \{T_2 \leq n\}.$$

Thus the conclusion (5) follows.
Note that

$$M_n \nearrow M \quad \text{and} \quad N_n \nearrow N \quad \text{as } n \to \infty.$$

Thus, under the assumption that $S_n \to S$ in law, (6) and (7) follow respectively from (4) and (5). $\square$

Applications

As applications of Theorems 1 and 3 above, the next two theorems provide inequalities for expected values.

Theorem 4 Let $\{X_n;\ n \geq 1\}$ be a sequence of independent symmetric **B**-valued random variables. Let $q:\ \mathbf{B}^{\infty} \to \overline{\mathbb{R}}_{+} = [0, \infty]$ be a measurable quasiconvex function (i.e., (1) holds). If $\varphi:\ \mathbb{R}_{+} \to \mathbb{R}_{+}$ is nondecreasing function and $\varphi^{-}(x) = \sup_{y<x} \varphi(y)$, then we have

$$\mathbb{E}\varphi(M_n) \leq 2\mathbb{E}\varphi(\alpha S_n), \quad n \geq 1,$$

$$\mathbb{E}\varphi^{-}(M) \leq 2 \liminf_{n\to\infty} \mathbb{E}\varphi(\alpha S_n),$$

$$\{S_n;\ n \geq 1\} \text{ is stochastically bounded} \Longleftrightarrow M < \infty \text{ a.s.}$$

Moreover if $S_n \to S$ in law, then we have

$$\mathbb{E}\varphi(S) \leq \mathbb{E}\varphi(M) \leq 2\mathbb{E}\varphi(\alpha S).$$

Applications

Theorem 5 Let $\{X_n;\ n \geq 1\}$ be a sequence of independent **B**-valued random variables. Let $q:\ \mathbf{B}^\infty \to \overline{\mathbb{R}}_+ = [0, \infty]$ be a measurable quasiconvex and subadditive even function. Let $\varphi:\ \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing function with $\varphi(2t) \leq K\varphi(t)$ for $t \geq a$, where $K \geq 1$ and $a \geq 0$ are constants depending only on φ. Suppose that $M < \infty$ a.s. Then the following statements are equivalent:

$$\mathbb{E}\varphi(M) < \infty, \tag{9}$$

$$\sup_{n \geq 1} \mathbb{E}\varphi\left(S_n\right) < \infty, \tag{10}$$

$$\mathbb{E}\varphi(N) < \infty. \tag{11}$$

Moreover if $S_n \to S$ in law, then (9)-(11) are each equivalent to:

$$\mathbb{E}\varphi(S) < \infty.$$

Applications

Let $p > 0$. For real numbers $x_1, x_2, ..., x_n, ...$, let

$$q\left(x_1, x_2, \cdots, x_n, \cdots\right) = \sup_{n \geq 2} \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} |x_i - x_j|^p .$$

Then clearly the function q satisfies the conditions (3) and (8) with

$$\alpha = \begin{cases} 1, & \text{if } p \geq 1 \\ 2^{1-p}, & \text{if } 0 < p < 1 \end{cases}$$

and

$$\beta = \begin{cases} 2^{p-1}, & \text{if } p \geq 1 \\ 1, & \text{if } 0 < p < 1. \end{cases}$$

Applications

Let $\{X, X_n;\ n \geq 1\}$ be a sequence of real-valued independent and identically distributed (i.i.d.) random variables. We now use the function q to obtain the Kolmogrov Strong Law of Large Numbers (SLLN) for the general Gini's mean difference

$$\frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} |X_i - X_j|^p$$

which is a well-known **U**-statistic for unbiased estimation of the dispersion parameter $\mathbb{E}|X_1 - X_2|^p$. For the case $p = 1$, Li, Rao, and Tomkins (*J. Multivariate Anal.* **78** (2001), 191-217) proved that

Applications

$$\limsup_{n\to\infty} \frac{2}{n(n-1)} \sum_{1\le i<j\le n} |X_i - X_j| < \infty \text{ a.s.}$$

if and only if

$$\mathbb{E}|X| < \infty.$$

In either case, we have

$$\lim_{n\to\infty} \frac{2}{n(n-1)} \sum_{1\le i<j\le n} |X_i - X_j| = \mathbb{E}\,|X_1 - X_2| \text{ a.s.}$$

The “if part” of this result is covered in R. Helmers, P. Janssen, and R. Serfling (*Probab. Theory Relat. Fields* **79** (1988), 75-93).

Applications

As an application of Theorem 1, the Kolmogrov SLLN for the general Gini's mean difference is presented in the following theorem.

Theorem 6 Let $p > 0$. Let $\{X, X_n;\ n \geq 1\}$ be a sequence of **B**-valued i.i.d. random variables. Then

$$\limsup_{n\to\infty} \frac{2}{n(n-1)} \sum_{1\leq i<j\leq n} \|X_i - X_j\|^p < \infty \quad \text{a.s.}$$

if and only if

$$\mathbb{E}\|X\|^p < \infty.$$

In either case, we have

$$\lim_{n\to\infty} \frac{2}{n(n-1)} \sum_{1\leq i<j\leq n} \|X_i - X_j\|^p = \mathbb{E}\,\|X_1 - X_2\|^p \quad \text{a.s.}$$

Applications

Theorem 5 can be used to study the integrability of

$$G_p \triangleq \sup_{n\geq 2} \frac{2}{n(n-1)} \sum_{1\leq i<j\leq n} \|X_i - X_j\|^p .$$

We have the following result.

Theorem 7 Let $p > 0$ and $q > 0$. Let $\{X, X_n;\ n \geq 1\}$ be a sequence of **B**-valued i.i.d. random variables. Then

$$\mathbb{E}G_p^q < \infty$$

if and only if

$$\begin{cases} \mathbb{E}|X|^p < \infty, & \text{if } 0 < q < 1 \\ \mathbb{E}\left(|X|^p \log(1+|X|)\right) < \infty, & \text{if } q = 1 \\ \mathbb{E}|X|^{pq} < \infty, & \text{if } q > 1. \end{cases}$$

Applications

Motivated by the results obtained by Li, Qi, and Rosalsky (JTP **24** (2011), 1130-1156), we introduce a new type strong law of large numbers as follows.

Definition *Let* $0 < p < 2$ *and* $0 < q < \infty$. *Let* $\{X_n;\ n \geq 1\}$ *be a sequence of independent copies of a* $\mathbf{B}$*-valued random variable* X. *We say* X *satisfies the* (p, q)*-type strong law of large numbers (and write* $X \in SLLN(p, q)$*) if*

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{\|S_n\|}{n^{1/p}} \right)^q < \infty \quad \text{a.s.}$$

Applications

By applying results obtained from the new versions of the classical Lévy, Ottaviani, and Hoffmann-Jørgensen (1974) inequalities proved by Li and Rosalsky (2013) and by using techniques developed by Hechner (PhD Thesis (2009)) and Hechner and Heinkel (JTP **23** (2010), 509-522), Li, Qi, and Rosalsky (TAMS **368** (2016), 539-561) have obtained sets of necessary and sufficient conditions for $X \in SLLN(p, q)$ for the six cases: $1 \leq q < p < 2$, $1 < p = q < 2$, $1 < p < 2$ and $q > p$, $q = p = 1$, $p = 1 < q$, and $0 < p < 1 \leq q$. The necessary and sufficient conditions for $X \in SLLN(p, 1)$ have been discovered by Li, Qi, and Rosalsky (JTP **24** (2011), 1130-1156). Versions of above results in a Banach space setting are also given.

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THANK YOU VERY MUCH!